MATH-232 Final Exam, Solutions

Exercise 1.

(a) *How many permutations of the letters of the word 'statistics' give a different sequence of letters?*

(b) *Among 200 students, what is the probability that at least two students have the same birthday (for example, August 14, without the year mattering) and the same eye color? Note: assume that the birthdays are independently and uniformly distributed among 365 days and that the eye colors are independently and uniformly distributed among blue, green, brown.*

Solution.

(a) (1 pt) Correct answer: $\frac{10!}{3!3!2!}$.

(b) (1 pt) Correct answer: $1 - \frac{1095!}{(1095-200)1095^200}$.

The sample space can be written as $\Omega = \{p_{i,j} = (i,j) : i \in [365], j \in [3]\}^{200}$, and each of these possibilities has probability $(365 \times 3)^{-200} = 1095^{-200}$. We compute firstly the probability of the complementary event $A = \{(p_{i_1,j_1}, ..., p_{i_n,,j_n}) : p_{i_1,j_1} \neq ... \neq p_{i_n,,j_n}\}$, i.e. the event that each pair of students have either different birthdays or different eye colors. There are $\frac{1095!}{(1095-n)!}$ ways this can happen, so the overall probability is $\frac{1095!}{(1095-n)!1095^{200}}$. The probability that at least two students have the same birthday and same eye color is thus $1 - \frac{1095!}{(1095-n)!1095^{200}}$. □

Exercise 2. *Assume that a person infected by COVID transmits the virus to another person during a meeting with probability (i)* $1/2$ *if none wears a mask, (ii)* $1/10$ *if only one person wears a mask (irrespective of which), and (iii)* $1/100$ *if both wear masks. Assume that people wear masks independently with probability* $1/2$*. Given that Alice transmitted the virus to Bob during a meeting, what is the probability that Bob wore a mask during that meeting? Note: we do not know if Alice wore a mask.*

Solution. Correct answer: $\frac{11}{71}$ (2 pt).
Write Bayes formula correctly (1 pt).
Write theorem of total probability correctly (0.5pt).

Let B be the event "Bob wore a mask" and A be the event "Alice wore a mask". Moreover, let I be the event "the virus was transmitted". Using Bayes rule we write

$$\mathbb{P}(I|B) = \frac{\mathbb{P}(I|B)\mathbb{P}(B)}{\mathbb{P}(I)}. \quad (1)$$

$$\mathbb{P}(I|B) = \mathbb{P}(I|B,A)\mathbb{P}(A) + \mathbb{P}(I|B,A^C)\mathbb{P}(A^C) = \frac{1}{2}\frac{1}{100} + \frac{1}{2}\frac{1}{10} = \frac{11}{200};$$
$$\mathbb{P}(I) = \mathbb{P}(I|B,A)\mathbb{P}(B,A) + \mathbb{P}(I|B,A^C)\mathbb{P}(B,A^C) + \mathbb{P}(I|B^C,A)\mathbb{P}(B^C,A) + \mathbb{P}(I|B^C,A^C)\mathbb{P}(B^C,A^C)$$
$$= \frac{1}{4}(\frac{1}{100} + \frac{2}{10} + \frac{1}{2}) = \frac{71}{400}$$
$$\mathbb{P}(B) = \frac{1}{2}.$$

By plugging the numbers in (1), one obtains the result. □

Exercise 3. *Let $X \sim \mathcal{N}(0,1)$ and $Y = X^2$.*

(a) *Are X and Y uncorrelated?*

(b) *Are X and Y independent?*

Solution.

(a) (1 pt) Correct answer: Yes
$E[XY] = E[X^3] = 0$.

(b) (1 pt) Correct answer: No
Y is a deterministic function of X. □

Exercise 4. *Let $X_1, \dots, X_n$ be i.i.d. such that X_i equals 0 with probability 1/2 and π with probability 1/2, for all $i \in [n]$. Show that $\frac{1}{n}\sum_{i=1}^n X_i$ tends in probability to a scalar x, and find the value for x. Note: you cannot invoke theorems from the class, you need to prove the statement in full (possibly using the formulaire).*

Recall: Y_n tends to Y in probability if for any $\epsilon > 0$, $\mathbb{P}(|Y_n - Y| > \epsilon) \overset{n\to\infty}{\longrightarrow} 0$.

Solution. $x = \frac{\pi}{2}$ (0.5 pt).
Let $S_n = \frac{1}{n}\sum_{i=1}^n X_i$, and notice $E[S_n] = \frac{\pi}{2}$ and $\mathrm{Var}(S_n) = \frac{\pi^2/4}{n}$. Then by Markov inequality for any $\epsilon > 0$:
$\mathbb{P}(|S_n - E[S_n]| > \epsilon) = \mathbb{P}((S_n - E[S_n])^2 > \epsilon^2) \leq \frac{E(S_n - E[S_n])^2}{\epsilon^2} = \frac{Var(S_n)}{\epsilon^2} \to 0$ for $n \to \infty$.

Proof correct applying directly Chebychev (1 pt) □

Exercise 5. *Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)$. Can you find a 2×2 matrix A such that AX has i.i.d. $\mathcal{N}(0,1)$ components? Disprove or provide A.*

Solution. Correct answer: $A = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 \end{bmatrix}$

We need to find A such that $A \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} A^T = \mathbb{I}_2$. One can find matrix A through eigenvector decomposition. □

Exercise 6. *Let $Y_1, \dots, Y_n$ be i.i.d. such that Y_i equals 1 with probability p and -1 with probability $1-p$, for all $i \in [n]$.*

(a) *Find an estimator of p using the method of moments.*

(b) *What is the MSE of your estimator in (a)?*

Recall: the MSE of an estimator $\hat{\theta}$ for a parameter θ is $\mathbb{E}(\hat{\theta} - \theta)^2$.

(c) *Find an estimator of p using the maximum likelihood method.*

Solution.

(a) (1 pt) Correct answer: $\hat{p} = \frac{\bar{y}+1}{2}$.
$\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i = E[Y_i] = 2p - 1$

(b) (1 pt) Correct answer: $MSE = \frac{p(1-p)}{n}$.
$MSE(\hat{p}) = Var(\hat{p}) + bias(\hat{p})^2$, with $bias(\hat{p})^2 = 0$ and $Var(\hat{p}) = \frac{p(1-p)}{n}$.

(c) (1 pt) Correct answer: $\hat{p} = \frac{\bar{y}+1}{2}$.
The log-likelihood is given by $l(p) = n\bar{y}\log(p) + \frac{1}{2}(n - n\bar{y})(\log(p) + \log(1-p))$. Solve $l'(p) = 0$ and $l''(p) < 0$. □

Exercise 7. *Let $Y_1, \dots, Y_n$ be i.i.d. $\mathcal{N}(\mu, 4)$ for some $\mu \in \mathbb{R}$.*

(a) *Find the values of a and b for which $(\sum_{i=1}^n Y_i/n - a)/b$ is distributed as $\mathcal{N}(0,1)$.*
Note: a, b may depend on μ, n.

(b) *Using this pivot, give an equi-tailed two-sided confidence interval for μ with confidence level $\alpha = 0.05$ (i.e. $\alpha_L = \alpha_U = \alpha/2$) in the special case where $\sum_{i=1}^n Y_i/n = 1$ and $n = 100$. Note: the formulaire contains Gaussian quantiles.*

Solution.

(a) (1 pt) Correct answer: $a = \mu$, $b = \frac{2}{\sqrt{n}}$.
$a = E[\frac{1}{n}\sum_{i=1}^n Y_i] = \mu$ and $b = \sqrt{Var(\bar{Y})} = \frac{2}{\sqrt{n}}$

(b) (1 pt) Correct answer: $(L, U) = (1 - \frac{1.96}{5}, 1 + \frac{1.96}{5})$.
$(L, U) = (\bar{Y} - \frac{\sigma}{\sqrt{n}} z_{0.975}, \bar{Y} - \frac{\sigma}{\sqrt{n}} z_{0.025})$, where $\sigma = 2$, $\bar{Y} = 1$ and $n = 100$.

□

Exercise 8. *Consider the binary hypothesis test where a single random variable Y is observed. Under H_0, the random variable Y takes values $\{0, 1, 2\}$ with probabilities $\{1/4, 1/2, 1/4\}$ respectively, and under H_1, the random variable Y takes values $\{1, 2, 3\}$ with probabilities $\{1/4, 1/2, 1/4\}$ respectively.*

(a) *Give a test (i.e., define the set of values of Y for which you accept, or reject, H_0) such that the false negative probability is minimal under the constraint that the false positive probability is at most $1/3$. Note: the question is symmetrical if you exchange the role of the false positive and false negative probabilities, so it is ok if you flip the two definitions.*

(b) *Same question as in (a) under the constraint that the false positive probability is at most $1/5$.*

Solution.

(a) (1 pt) Correct answer: Reject H_0 if $Y > 1$, accept H_0 otherwise.

(b) (1 pt) Correct answer: Reject H_0 if $Y > 2$, accept H_0 otherwise. □